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SUPPRESSION OF LIMIT CYCLES IN A CLASS OF NON-LINEAR SYSTEMS BY DISTURBANCE OBSERVERS

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1. INTRODUCTION

Major goals of controllers in feedback systems are to stabilize systems (plants) and have them perform satisfactorily. A well-designed feedback system usually has properties such as: (1) its output is insensitive to disturbances; (2) its performance is insensitive to model uncertainties and variations; (3) it behaves more or less like a linear system. In particular, these properties are achieved when the controller gain is increased (high gain control), as long as the system remains stable; see, e.g., references [1–6].

Systems that are linear or operate around a certain operating point can usually achieve satisfactory performance by linear controllers in single-degree-of-freedom configurations. Such controllers, however, may not be adequate if there are (significant) non-linearities in the system. Non-linearities can introduce a variety of interesting and sometimes undesirable phenomena in systems. One such phenomenon is the limit cycle behavior. Limit cycles are often considered undesirable and thus, means are sought to suppress them. In the past decades, researchers have devised techniques to suppress limit cycles; see, e.g., references [7–10].

In this note, it is shown that an effective means of suppressing limit cycles in non-linear systems is the application of disturbance observers. Disturbance observers are useful tools that were originally proposed in references [11, 12] as means of estimating disturbances to linear systems and cancelling them subsequently. Later, the theory of disturbance observers was advanced in reference [13]. Presently, disturbance observers are successfully used in achieving robust stability and performance in motion control systems, for instance, in controlling robotics systems, high-speed machining systems, (micro) positioning systems, disk drives; see, e.g., references [14–17] and the references therein. It appears that disturbance observers are mostly designed for linear systems. There are only a few pieces of work where the application of disturbance observers to non-linear systems is reported; see references [18–20]. The present note illustrates a novel application of disturbance observers to a class of non-linear systems that exhibit the limit cycle behavior.

The organization of the note is as follows. In section 2, the class of non-linear systems to be studied is presented. The non-linearity in a system of this class has the

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property that its output can be decomposed as the summation of the outputs of a stable single-input single-output (SISO) linear time-invariant system and a bounded function of time. In section 3, disturbance observers are designed to estimate the effects of non-linearities that will be cancelled subsequently. The designed disturbance observers are thus able to make the non-linear systems under consideration behave linearly and, for instance, be free of the limit cycle behaviour. An example is presented to illustrate how well disturbance observers can suppress limit cycles.

2. NON-LINEAR FEEDBACK SYSTEMS

In this section, a class of SISO unity feedback non-linear systems is introduced. A member of this class is denoted by $S(N, P)$ and is shown in Figure 1. In this system:

Signals: The input $r \in L_{\infty e}(\mathbb{R}_+)$, the output $y \in L_{\infty e}(\mathbb{R}_+)$, and the measurement noise $\xi \in L_\infty(\mathbb{R}_+)$, ($L_{\infty e}(\mathbb{R}_+)$ denotes the extended L_∞-space on $\mathbb{R}_+$; see, e.g., references [3, 21] for the definition of such spaces); hereafter, $(\mathbb{R}_+)$ is deleted in the notation of spaces.

Plant: $P: L_{\infty e} \to L_{\infty e}$ represents a (possibly unstable) SISO linear time-invariant plant.

Non-linearity: $N: L_{\infty e} \to L_{\infty e}$ represents a SISO non-linear time-varying system that can be decomposed as

$$N = H + \Phi, \tag{1}$$

where

(i) $H: L_{\infty e} \to L_{\infty e}$ is a stable SISO linear time-invariant system;
(ii) $\Phi: L_{\infty e} \to L_\infty$ is a SISO time-varying non-linearity, whose output is given by

$$d(t) := (\Phi u)(t) = \phi(u(t), t), \tag{2}$$

for all $t \geqslant 0$ and $u \in L_{\infty e}$, where $\phi: \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$. It is assumed that $d \in L_\infty$ (that is, it is *bounded*) for any input $u \in L_{\infty e}$.

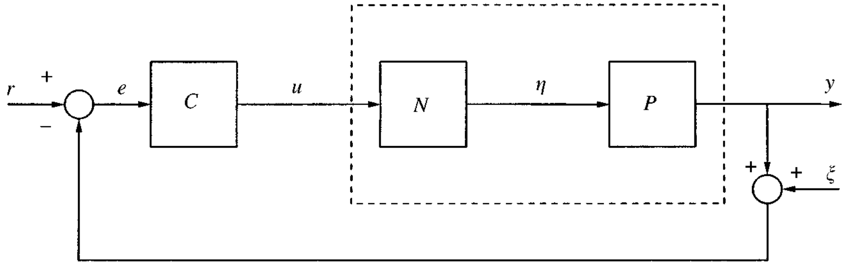

Figure 1. The non-linear feedback system $S(N, P)$. The non-linearity N and the linear plant P form an integral part of the system, such that η cannot be measured. The controller C achieves the bounded-input bounded-output stability of the system.

By equations (1) and (2), the output of the non-linearity N is the summation of the outputs of a stable linear time-invariant system and a bounded function of time, i.e.,

$$\eta(t) = (Nu)(t) = (Hu)(t) + d(t), \tag{3}$$

for all $t \geqslant 0$, where $(Hu)(t) = H(t) * u(t)$ is the convolution of the impulse response $H(t)$ of the linear system H and $u(t)$.

Controller: $C: L_{\infty e} \to L_{\infty e}$ is a SISO linear or non-linear controller that at least achieves the bounded-input bounded-output (BIBO) stability of the system $S(N, P)$. □

Remarks: (1) The non-linearity N and the linear plant P form an integral part of the system $S(N, P)$. Thus, the output of N, denoted by η is Figure 1, cannot be measured.

(2) The outputs of several non-linearities of interest can be represented by equation (3). For instance, let N be a SISO time-varying non-linearity whose graph lies between or on two curves G_+ and G_- for all instances of time (see Figure 2). Furthermore, suppose that there exists a line of *non-zero* and *finite* slope K, passing through the origin and lying in the region between the curves G_+ and G_-, such that $G_+(u) - Ku$ and $Ku - G_-(u)$ are finite for all $u \in \mathbb{R}$. Thus, the linear part of N is the constant gain K and its outputs is

$$\eta(t) = (Nu)(t) = Ku(t) + d(t), \tag{4}$$

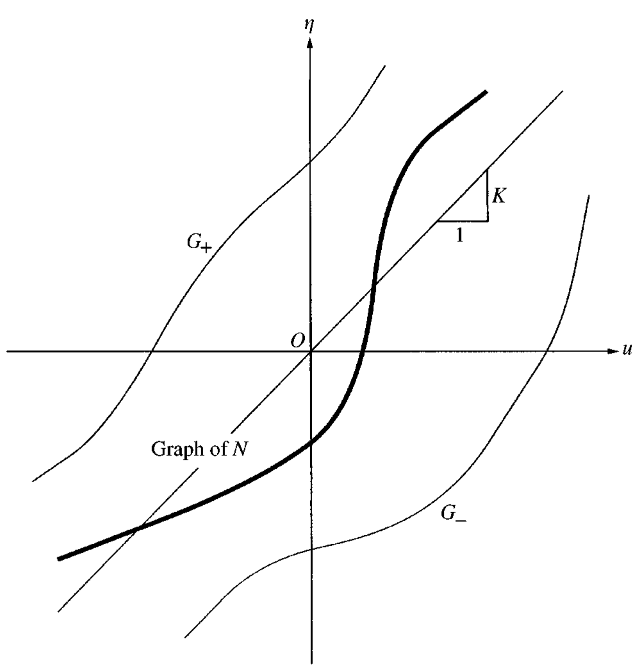

Figure 2. An example of a single-input single-output non-linearity N, which can be decomposed as a constant gain K and a non-linearity Φ of bounded output.

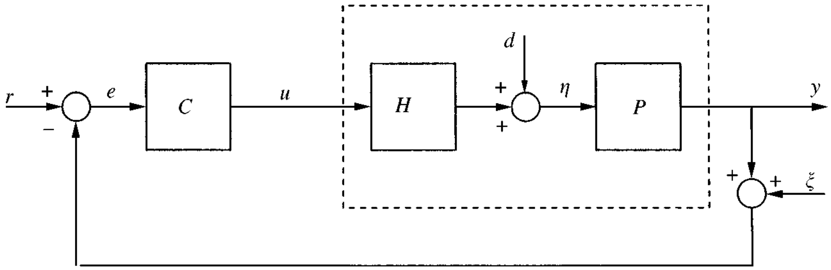

Figure 3. The feedback system $S(H, P)$. This system is equivalent to $S(N, P)$ from the input–output point of view. The system H is linear.

for all $t \geqslant 0$ and $u \in L_{\infty e}$, where

$$\| d \|_\infty = \sup_{t \geqslant 0} |d(t)| \leqslant \max \{ \sup_{u \in \mathbb{R}} (G_+(u) - Ku), \sup_{u \in \mathbb{R}} (Ku - G_-(u)) \}, \tag{5}$$

Typical examples of non-linearities, whose graph lie between two curves such as G_+ and G_-, are dead-zone, non-linear amplifiers (see, e.g., reference [22]), and backlash; for these non-linearities G_+ and G_- are parallel lines. □

Having the non-linearity N satisfying the decomposition in equation (1), a system is introduced which is equivalent to $S(N, P)$ from the input–output point of view. This equivalent system is denoted by $S(H, P)$ and is depicted in Figure 3. In this system, P is the same as that in $S(N, P)$ and H is the linear part of N, introduced in equation (1). The input r and the measurement noise ξ in $S(H, P)$ are the same as those in $S(N, P)$, and d is the (bounded) output of the non-linearity Φ, introduced in equation (2). Having the system $S(H, P)$ so defined, C can be chosen to be a *linear* time-invariant controller, by which $S(H, P)$ will be a linear feedback system. In this case, C, H, and P can be represented by their respective transfer functions $C(s)$, $H(s)$, and $P(s)$.

The representation of the non-linear system $S(N, P)$ by the equivalent linear system $S(H, P)$ was first introduced in reference [6] for the multi-input multi-output (MIMO) case. Due to the linearity of $S(H, P)$, useful results can be established for this system, which in turn hold for $S(N, P)$. For instance, in reference [6]: (1) the set of all linear controllers C that achieve the BIBO stability of the system $S(H, P)$ (equivalently $S(N, P)$) is obtained; (2) it is shown that the non-linear effect of N in the system $S(N, P)$ can be reduced as the gain of the controller C is increased.

The linear representation $S(H, P)$ was later used in reference [23] to design controllers for the system $S(N, P)$ by the quantitative feedback theory (QFT). The system $S(H, P)$ has yet another useful property to be exploited in the next section.

3. LINEAR BEHAVIOR BY DISTURBANCE OBSERVERS

Representing the non-linear system $S(N, P)$ by the equivalent linear system $S(H, P)$ is of great advantage, because the non-linear effect of N in $S(N, P)$ appears

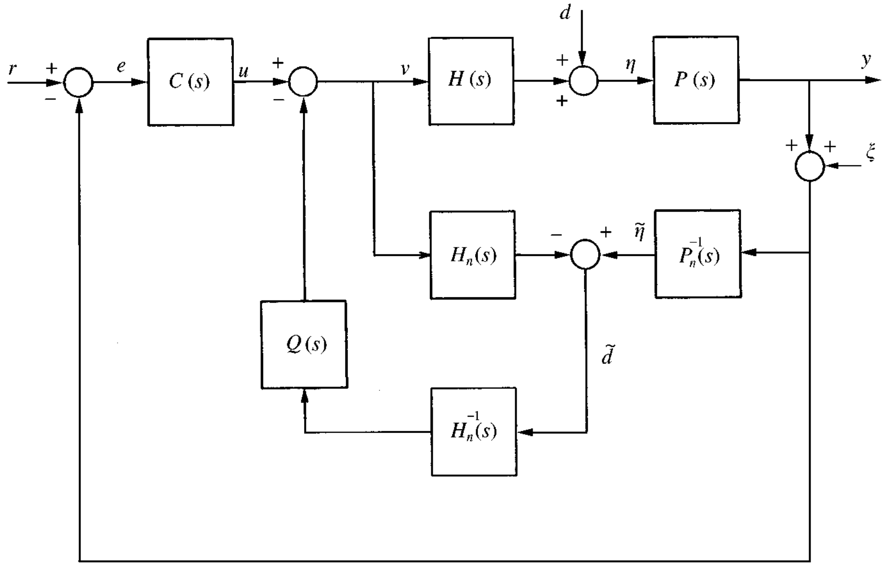

Figure 4. A disturbance observer is added to the system $S(H, P)$ (equivalently, $S(N, P)$) to estimate d, which is the non-linear effect of N in $S(N, P)$. The disturbance observer provides an estimate of d, denoted by $\tilde{d}$, to be cancelled subsequently.

as the bounded disturbance d in $S(H, P)$. Therefore, if one seeks to suppress the non-linear effect of N is $S(N, P)$, then one should design a controller that suppresses the effect of the disturbance d in $S(H, P)$. The latter can be achieved by a disturbance of observer that estimates d and cancels it subsequently. Therefore, the goal in this section is to design disturbance observers to make $S(N, P)$ behave like a linear system and, for instance, be free of the limit cycle behavior.

A disturbance observer added to the system, $S(H, P)$ is shown in Figure 4. In this figure, $H_n(s)$ and $P_n(s)$, respectively, represent the nominal transfer functions (mathematical models) corresponding to $H(s)$ and $P(s)$. The output of $P_n^{-1}(s)$, denoted by $\tilde{\eta}$, is close to η. Clearly, $\tilde{d}(t) := \tilde{\eta}(t) - (H_n v)(t)$, where $(H_n v)(t)$ denotes the convolution of the impulse response of $H_n(s)$ and $v(t)$, is an estimate of the disturbance $d(t)$ for all $t \geqslant 0$. In order to implement the disturbance observer, the filter $Q(s)$ is added to the system to make $Q(s)H_n^{-1}(s)P_n^{-1}(s)$ a realizable (at least a proper) transfer function, because $H_n^{-1}(s)$ and $P_n^{-1}(s)$ are often unrealizable. A successful design of a disturbance observer crucially depends on the design of $Q(s)$. Due to its crucial role, the design of $Q(s)$ has been extensively studied by researchers; see, e.g., references [11, 13, 14]. It turns out that $Q(s)$ should be a low-pass filter with the unity DC-gain. A typical form of $Q(s)$ is

$$Q(s) = \frac{\sum_{k=1}^{m-\rho} a_k (\tau s)^k + 1}{\sum_{k=1}^{m} a_k (\tau s)^k + 1}, \tag{6}$$

where ρ is at least equal to the summation of the relative degrees of $H_n(s)$ and $P_n(s)$ and a_k and τ are positive real numbers. A realizable implementation of the

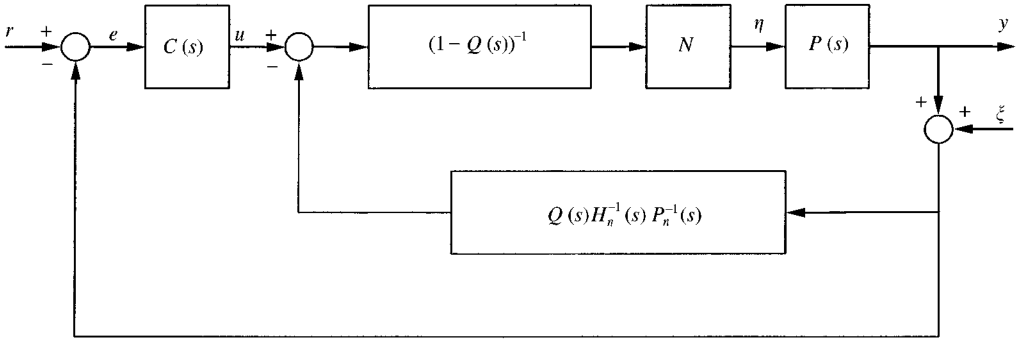

Figure 5. The feedback system $S(N, P; DOB)$. This system is $S(N, P)$ to which a disturbance observer is added.

disturbance observer for the system $S(N, P)$ (equivalently $S(H, P)$) is shown in Figure 5. The system in this figure is denoted by $S(N, P; DOB)$ to indicate a disturbance observer is added to $S(N, P)$.

Next, an example is presented to illustrate the efficacy of disturbance observers in suppressing limit cycles in a non-linear system.

3.1. EXAMPLE: A SYSTEM WITH BACKLASH

Backlash is an undesirable non-linearity that arises in gear systems. It usually slows down systems and can cause the limit cycle instability. Therefore, it is desirable to suppress the adverse effect of backlash in systems. In this example, it is shown that a disturbance observer can suppress limit cycles caused by backlash.

Consider the system $S(N, P)$ and let the linear plant P be an inverted pendulum driven by a motor and gear system, where the gears have backlash. A simplified model of the plant is

$$P(s) = \frac{1}{s^2 - 1}. \tag{7}$$

Let the gear backlash, denoted by N, be that shown in Figure 6. Clearly, the graph of N lies in the region between or on two parallel lines $G_+(u) = u + 0{\cdot}5$ and $G_-(u) = u - 0{\cdot}5$ for all $u \in \mathbb{R}$. Hence, N can be decomposed as that in equation (1), where the transfer function of the linear part is $H(s) = 1$ and Φ is a non-linearity whose output d is bounded by 0·5. Therefore, the non-linear system $S(N, P)$ can be represented by $S(H, P)$ in Figure 2. Having the representation $S(H, P)$, it can be easily verified that for the controller

$$C(s) = \frac{K_c(s + 2)}{s + 3}, \tag{8}$$

where $K_c = 10$, the system $S(H, P)$ is the BIBO stable, and so is $S(N, P)$. (By the Routh–Hurwitz test, it can be shown that $S(H, P)$ is the BIBO stable for any $K_c > 1{\cdot}5$.)

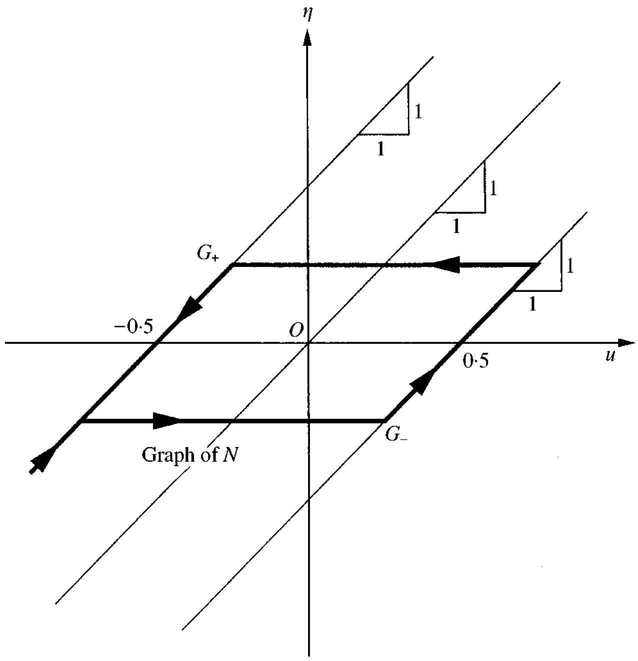

Figure 6. A backlash non-linearity N.

The response of the system $S(N, P)$ to the input

$$r(t) = \begin{cases} 0{\cdot}1, & 0{\cdot}1 \leqslant t \leqslant 0{\cdot}2, \\ 0, & 0{\cdot}2 < t, \end{cases} \tag{9}$$

in the absence of measurement disturbance ξ is plotted in Figure 7(a) and is designated by y_N. This response is a periodic function of time, which implies that the system has the limit cycle behavior. This behavior is anticipated, because there is a backlash in the system and the plant (inverted pendulum) is unstable. If there were no backlash in $S(N, P)$, i.e., $d \equiv 0$ in $S(H, P)$, then the system response to the same input r would have decayed to zero, as shown by y_L in Figure 7(a).

It is now shown that the non-linear effect of the backlash can be completely suppressed by a disturbance observer. The implementation of the disturbance observer is the same as $S(N, P;\, DOB)$ in Figure 5, with $H_n(s) = 1$ and $P_n(s) = P(s)$. In the implementation, the filter $Q(s)$ is chosen as

$$Q(s) = \frac{1}{(\tau s)^2 + 0{\cdot}5(\tau s) + 1}, \tag{10}$$

where $\tau = 0{\cdot}003$ s. The response of $S(N, P;\, DOB)$ is shown in Figure 7(a) and is designated by y_{DOB}. It is evident that y_L and y_{DOB} overlap. That is, the disturbance observer has successfully suppressed the limit cycles caused by the backlash. In

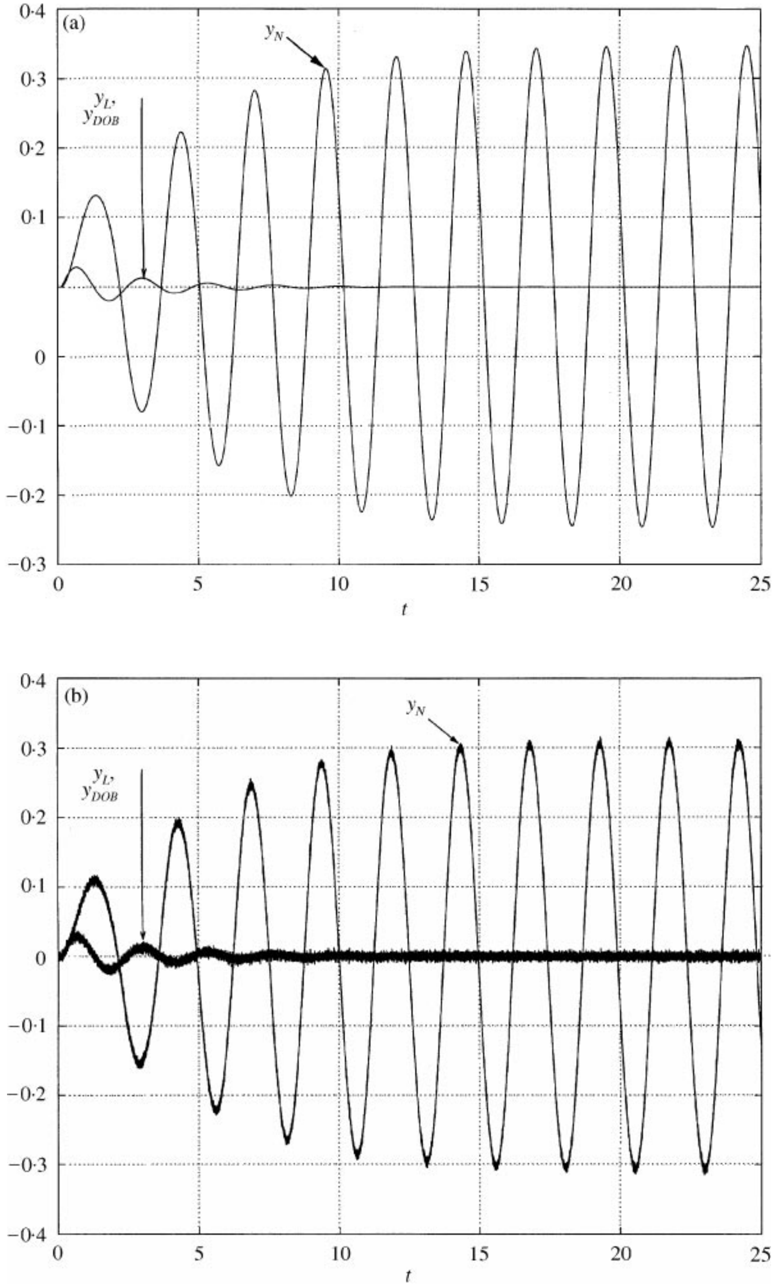

Figure 7. (a) Responses of the systems $S(N, P)$, the disturbance free ($d \equiv 0$) $S(H, P)$, and $S(N, P; DOB)$, denoted by y_N, y_L, and y_{DOB}, respectively, in the absence of measurement noise ξ. It is evident that y_L and y_{DOB} overlap and converge to zero. That is, the disturbance observer has suppressed the limit cycles caused by the backlash. (b) Responses of the systems $S(N, P)$, the disturbance free $S(H, P)$, and $S(N, P; DOB)$, denoted by y_N, y_L, and y_{DOB}, respectively, in the present of measurement noise ξ.

other words, the disturbance observer has made the system $S(N, P;\ DOB)$ behave linearly.

The effect of the measurement noise ξ on the performance of the system $S(N, P;\ DOB)$ is also studied. The noise ξ is chosen a band-limited white noise and is applied together with the input r in equation (9) to the systems $S(N, P)$, the disturbance free ($d \equiv 0$) $S(H, P)$, and $S(N, P;\ DOB)$ to, respectively, obtain responses y_N, y_L, and y_{DOB} in Figure 7(b). It is evident that the performance of $S(N, P;\ DOB)$ is not worse than those of $S(N, P)$ and the disturbance free $S(H, P)$.

4. CONCLUSIONS

In this note, a large class of SISO non-linear systems is considered. The non-linearity in a system of this class has the property that its output can be decomposed as the summation of the outputs of a stable SISO linear time-invariant system and a bounded function of time. Such non-linearities arise in many practical situations; for instance, dead-zone, backlash, and hysteresis non-linearities, to name a few. For the class of systems under consideration, disturbance observers are designed to estimate the effects of non-linearities and to cancel them subsequently. The designed disturbance observers are thus able to make the non-linear systems under consideration behave linearly and be free of the limit cycle behavior, as it is shown in an example. It is interesting to note that disturbance observers are linear systems, but yet are able to suppress the effect of non-linearities.

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